

# Painlevé III asymptotics of Hankel determinants for a perturbed Jacobi weight

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## Abstract

We study the Hankel determinants associated with the weight

$$w(x; t) = (1 - x^2)^\beta (t^2 - x^2)^\alpha h(x), \quad x \in (-1, 1),$$

where  $\beta > -1$ ,  $\alpha + \beta > -1$ ,  $t > 1$ ,  $h(x)$  is analytic in a domain containing  $[-1, 1]$  and  $h(x) > 0$  for  $x \in [-1, 1]$ . In this paper, based on the Deift-Zhou nonlinear steepest descent analysis, we study the double scaling limit of the Hankel determinants as  $n \rightarrow \infty$  and  $t \rightarrow 1$ . We obtain the asymptotic approximations of the Hankel determinants, evaluated in terms of the Jimbo-Miwa-Okamoto  $\sigma$ -function for the Painlevé III equation. The asymptotics of the leading coefficients and the recurrence coefficients for the perturbed Jacobi polynomials are also obtained.

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## 1 Introduction and statement of results

Let  $w(x; t)$  be the perturbed Jacobi weight

$$w(x; t) = (1 - x^2)^\beta (t^2 - x^2)^\alpha h(x), \quad x \in (-1, 1), \quad (1.1)$$

where  $\beta > -1$ ,  $\alpha + \beta > -1$ ,  $t > 1$ , the function  $h(x)$  is analytic in a domain containing  $[-1, 1]$  and  $h(x) > 0$  for  $x \in [-1, 1]$ . We study the Hankel determinants

$$D_n[w(x; t)] = \det(\mu_{j+k})_{j,k=0}^{n-1}, \quad (1.2)$$

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where  $\mu_i$  is the  $i$ -th moment of  $w(x; t)$ , namely,

$$\mu_i = \int_{-1}^1 x^i w(x; t) dx, \quad i = 0, 1, \dots.$$

The Hankel determinants possess the well-known multiple integral representation [21, (2.2.11)],

$$D_n[w(x; t)] = \frac{1}{n!} \int_{[-1, 1]^n} \prod_{i=1}^n w(x_i, t) \prod_{i < j} |x_i - x_j|^2 \prod_{i=1}^n dx_i.$$

Via the above integral representation, the Hankel determinants are closely related to various fundamental quantities in random matrix theory, such as the partition function, the gap probability of eigenvalues and the moment generating function of a certain random variable associated with the random matrix ensemble; see [20]. For example, in the Jacobi unitary ensemble corresponding to the weight  $w_1(x) = (1 - x^2)^\alpha$ , it is well-known that the probability distribution of the largest eigenvalues is

$$\begin{aligned} P_n(\lambda_{\max} < s) &= \frac{1}{n! D_n[w_1(x)]} \int_{[-1, s]^n} \prod_{i=1}^n (1 - x_i^2)^\alpha \prod_{i < j} |x_i - x_j|^2 \prod_{i=1}^n dx_i \\ &= \frac{\left(\frac{1+s}{2}\right)^{n^2+2n\alpha}}{n! D_n[w_1(x)]} \int_{[-1, 1]^n} \prod_{i=1}^n (1 + x_i)^\alpha (\varpi(s) - x_i)^\alpha \prod_{i < j} |x_i - x_j|^2 \prod_{i=1}^n dx_i, \end{aligned}$$

with  $\varpi(s) = \frac{3-s}{1+s}$ ; see [20]. Also, there is the remarkable Tracy-Widom formula for the large- $n$  asymptotics of the distribution of the extreme eigenvalues near the hard edge,

$$\lim_{n \rightarrow \infty} -\frac{d}{ds} \ln P_n(\lambda_{\max} < 1 - \frac{s}{2n^2}) = \frac{\sigma_{\text{JM}}(s)}{s}; \quad (1.3)$$

see [22], where  $\sigma_{\text{JM}}$  satisfies the Jimbo-Miwa-Okamoto  $\sigma$ -form of the Painlevé III equation ([17, (3.13)])

$$(s\sigma_{\text{JM}}'')^2 + \sigma_{\text{JM}}'(\sigma_{\text{JM}} - s\sigma_{\text{JM}}')(4\sigma_{\text{JM}}' - 1) - \alpha^2 \sigma_{\text{JM}}'^2 = 0 \quad (1.4)$$

and the boundary conditions

$$\sigma_{\text{JM}}(s) \sim \frac{1}{4^{\alpha+1} \Gamma(1+\alpha) \Gamma(2+\alpha)} s^{1+\alpha}, \quad s \rightarrow 0; \quad \sigma_{\text{JM}}(s) \sim \frac{s}{4} - \frac{\alpha}{2} \sqrt{s}, \quad s \rightarrow \infty. \quad (1.5)$$

It is worth noting that the Tracy-Widom formula (1.3) holds for a large family of unitary ensembles, including the modified Jacobi unitary ensemble associated with the weight

$$(1-x)^\alpha (1+x)^\beta h(x), \quad x \in (-1, 1), \quad \alpha > -1, \quad \beta > -1.$$

The phenomenon is termed universality in random matrix theory.

In [14], Forrester and Witte apply the Okamoto  $\tau$ -function theory to study the random matrix average for the Laguerre unitary ensemble

$$E(s, n) = \frac{1}{Z_n} \int_{[s, \infty)^n} \prod_{i=1}^n (x_i - s)^\beta x_i^\alpha e^{-x_i} \prod_{i < j} |x_i - x_j|^2 \prod_{i=1}^n dx_i,$$

where  $Z_n$  is the normalization constant. It is shown that the logarithmic derivative of the average  $E(s, n)$  satisfies the Jimbo-Miwa-Okamoto  $\sigma$ -form of the Painlevé V equation, with parameters depending on  $n$ . By taking the scaling  $s = \frac{t}{4n}$  and letting  $n \rightarrow \infty$ , it is found that the Painlevé V equation degenerates to a general Jimbo-Miwa-Okamoto  $\sigma$ -form of the Painlevé III equation. Thus the hard edge limiting average is obtained, generalizing the results of Tracy and Widom (1.3)-(1.5). The boundary conditions of the Painlevé III equation have also been studied in the follow-up paper [15] of the same authors.

Now we mention several weights closely related to (1.1). A decade ago, in [18, 19], Kuijlaars *et al.* considered the orthogonal polynomials associated with the weight

$$(1+x)^\alpha (1-x)^\beta h(x), \quad x \in (-1, 1),$$

which, in the case  $\alpha = \beta$ , is the weight (1.1) with  $t = 1$ . The main focus of [18] is to obtain the asymptotics of the orthogonal polynomials, including those of the recurrence coefficients, the leading coefficients and the Hankel determinants. In [19], the results find applications in random matrix theory, the Bessel limit kernel is obtained at the hard edge, and the kernel is independent of the perturbed analytic function  $h$ . Later, in [23], Vanlessen studies the orthogonal polynomials associated with the further generalized Jacobi-type weight with several singularities, of the form

$$(1+x)^\alpha (1-x)^\beta h(x) \prod_{\nu=1}^p |x - x_\nu|^{2\lambda_\nu}, \quad x \in (-1, 1),$$

where  $p$  is a fixed integer,  $-1 < x_1 < x_2 < \dots < x_p < 1$ ,  $2\lambda_\nu > -1$ ,  $\lambda_\nu \neq 0$ ,  $\alpha, \beta > -1$ , and  $h$  is real analytic and strictly positive on  $[-1, 1]$ . The asymptotics of the recurrence coefficients and the orthogonal polynomials associated are also obtained.

Very recently, Basor, Chen and Haq [1] study the Hankel determinants associated with the weight

$$\hat{w}(x, k) = (1 - x^2)^\beta (1 - k^2 x^2)^\alpha, \quad x \in (-1, 1), \quad \beta > -1, \alpha \in \mathbb{R},$$

which is a special case of the weight (1.1) with  $t = 1/k$ , and  $h = k^{2\alpha}$ . For  $n$  fixed, it is shown in [1], via the ladder operator method, that the finite Hankel determinant  $D_n[\hat{w}(x; k)]$  is the  $\tau$ -function of the Painlevé VI equation; see also [3] for applications of the ladder operator method. Large- $n$  asymptotics of the Hankel determinants are also obtained in [1] for fixed  $k$ .

In this paper, however, we focus on the asymptotics of the Hankel determinants, the leading coefficients, and the recurrence coefficients associated with the weight (1.1), in

the sense of a double scaling limit as  $n \rightarrow \infty$  and  $t \rightarrow 1$  when the algebraic singularity  $x = t$  approaches the hard edge  $x = 1$ .

Remarkable progress has been made in the study of the double scaling limit of Hankel determinants and Toeplitz determinants owing to the Riemann-Hilbert approach developed by Deift, Zhou *et al.* [8, 11, 12]. A series of questions and conjectures arose in the analysis of the Ising model (see [10]) and random matrices have been solved. For example, in [5, 9], Claeys, Deift and co-authors obtain Painlevé V asymptotics of Toeplitz determinants and Hankel determinants associated with the emergence of a Fisher-Hartwig singularity in the weight function. More recently, Claeys and Krasovsky [6] study a weight with merging Fisher-Hartwig singularities and again a Painlevé V function is involved to describe the transition between two different types of asymptotic behavior of the Toeplitz determinants.

Other types of singularities have also been encountered. In [2], Brightmore, Mezzadri and Mo consider the asymptotics of the partition function associated with the Gaussian weight perturbed by an essential singularity and they get Painlevé III type asymptotics. In [24, 25], Xu, Dai and Zhao also obtain Painlevé III type asymptotics of the Hankel determinants associated with the Laguerre weight with an essential singularity at the hard edge. In the double scaling limit of Hankel determinants, the appearance of Painlevé functions is of particular interests; cf., e.g., [4, 7, 16, 27]. The reader is referred to the comprehensive survey paper [10] for the historic background and updated results on the theory of Hankel determinants and Toeplitz determinants with applications in the Ising model.

In the preceding papers [26, 28], the authors have studied the transition asymptotics of the eigenvalue correlation kernel for the perturbed Jacobi unitary ensemble defined by the perturbed Jacobi weight given in (1.1), varying from the Bessel kernel  $\mathbb{J}_\beta$  to  $\mathbb{J}_{\alpha+\beta}$  as the parameter  $t$  varies in  $(1, d]$  for a fixed  $d > 1$ . A new class of universal behavior at the edge of the spectrum for the modified Jacobi ensemble is obtained and described in terms of the generalized Painlevé V equation, which in this case is equivalent to the Painlevé III equation after a Möbius transformation.

In the present paper, we focus on the asymptotic approximations of the Hankel determinants, the leading coefficients, and the recurrence coefficients of the polynomials orthogonal with respect to the weight (1.1), in the sense of a double scaling limit as  $n \rightarrow \infty$  and  $t \rightarrow 1$ . To simplify our discussion, we consider the even weight function (1.1) by assuming  $h$  is even, then we have the recurrence relation

$$z\pi_n(z) = \pi_{n+1}(z) + b_{n-1}^2\pi_{n-1}(z) \quad (1.6)$$

for monic orthogonal polynomials with respect to the perturbed Jacobi weight.

## 1.1 Statement of results

To state the main results, we need the Jimbo-Miwa-Okamoto  $\sigma$ -form of the Painlevé III equation

$$(s\sigma''_{\text{JM}})^2 + \sigma'_{\text{JM}}(\sigma_{\text{JM}} - s\sigma'_{\text{JM}})(4\sigma'_{\text{JM}} - 1) - c_2\sigma_{\text{JM}}^2 - c_1\sigma'_{\text{JM}} - c_0 = 0; \quad (1.7)$$

cf. [17, (3.13)], where  $c_2 = (\alpha + \beta)^2$ ,  $c_1 = -\frac{1}{2}\beta(\alpha + \beta)$  and  $c_0 = \frac{\beta^2}{16}$ .

Our first result is on the Painlevé III asymptotic approximations of the Hankel determinants, in terms of the Jimbo-Miwa-Okamoto  $\sigma$ -notation.

**Theorem 1.** *Let  $\beta > -1$ ,  $\alpha + \beta > -1$ ,  $t > 1$ , and  $D_n(t)$  be the Hankel determinants given in (1.2) corresponding to the weight  $w(x; t)$  in (1.1). As  $n \rightarrow \infty$  and  $t \rightarrow 1$ , we have the following asymptotic expansion*

$$\begin{aligned} \ln D_n(t) = & (n + \alpha + \beta)V_0 - \alpha \ln h(t) - \beta \ln h(1) + \frac{1}{2} \sum_{k=1}^{\infty} kV_k^2 + \left[ (\alpha + \beta)^2 - \frac{1}{4} \right] \ln \frac{n}{4} \\ & - [n^2 + 2n(\alpha + \beta) + 1] \ln 2 + \left[ n + \alpha + \beta + \frac{1}{2} \right] \ln 2\pi + 2 \ln \frac{G(\frac{1}{2})}{G(\alpha + \beta + 1)} \\ & - \frac{\alpha^2}{2} \ln t + \frac{1}{2}(n \ln \varphi(t))^2 - 4 \int_0^{4n \ln \varphi(t)} \frac{\sigma_{\text{JM}}(\frac{s^2}{16})}{s} ds + o(1), \end{aligned} \quad (1.8)$$

where  $V_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ki\theta} \ln h(\cos \theta) d\theta$ ,  $k = 0, 1, \dots$ ,  $G(z)$  is the Barnes  $G$ -function defined in (5.40),  $\varphi(t) = t + \sqrt{t^2 - 1}$ , the Jimbo-Miwa-Okamoto  $\sigma$ -function  $\sigma_{\text{JM}}$  is analytic on  $(0, +\infty)$  and solves the equation (1.7) with the boundary conditions

$$\sigma_{\text{JM}}(s) = \frac{\beta}{4(\alpha + \beta)}s + C(\alpha, \beta) \left(\frac{s}{4}\right)^{1+\alpha+\beta} + O(s^{2+\alpha+\beta}) + O(s^2) \quad \text{as } s \rightarrow 0 \quad (1.9)$$

and

$$\sigma_{\text{JM}}(s) = \frac{1}{4}s - \frac{\alpha}{2}\sqrt{s} + \frac{1}{4}(\alpha^2 + 2\alpha\beta) - \frac{\alpha(\beta^2 - \frac{1}{4})}{4\sqrt{s}} + O(s^{-1}) \quad \text{as } s \rightarrow \infty, \quad (1.10)$$

with

$$C(\alpha, \beta) = \frac{\alpha\Gamma(1 - \alpha - \beta)\Gamma(\beta + 1)}{(\alpha + \beta)\Gamma(1 - \alpha)\Gamma(\alpha + \beta + 2)\Gamma(\alpha + \beta + 1)}.$$

*Remark 1.* For  $\alpha + \beta = 0$ , the condition (1.9) is simplified to

$$\sigma_{\text{JM}}(s) = \frac{s}{4} + O(s^2) \quad \text{as } s \rightarrow 0.$$

*Remark 2.* For  $\beta = 0$ , the theorem is reduced to the celebrated Tracy-Widom formula for the large- $n$  asymptotic distribution of the largest eigenvalues near the hard edge; cf. (1.3)-(1.5), see also [22].

Our second result is on the transition asymptotics of the leading coefficients of the corresponding orthonormal polynomials, where the Jimbo-Miwa-Okamoto  $\sigma$ -function is also involved.

*Theorem 2.* Let  $n \rightarrow \infty$  and  $t \rightarrow 1$ , then we have the asymptotic approximation of the leading coefficient of the orthonormal polynomial of degree  $n$  with respect to (1.1)

$$\frac{\gamma_n}{2^n} = \frac{1}{\sqrt{\pi} D_t(\infty)} \left( 1 + 2\sqrt{2} \left( \frac{\alpha}{2} + \frac{q}{s} \right) \sqrt{t-1} + c_n \varphi(t)^{-2\alpha} + O(t-1) + O\left(\frac{s}{n^2}\right) \right), \quad (1.11)$$

where  $\varphi(t) = t + \sqrt{t^2 - 1}$ ,  $D_t(\infty) = 2^{-(\alpha+\beta)} e^{-\frac{1}{2}V_0} \varphi(t)^\alpha$ ,  $V_0 = \frac{1}{2\pi} \int_0^{2\pi} \ln h(\cos \theta) d\theta$ ,  $q(s) = 4\sigma_{\text{JM}}(\frac{s^2}{16}) - \frac{s^2}{16} - (\alpha + \beta)^2 + \frac{1}{4}$ ,  $s = 4n \ln \varphi(t)$ ,  $c_n$  is independent of  $t$  such that  $c_n = O(n^{-2})$ , and the error term  $O\left(\frac{s}{n^2}\right)$  is uniform for  $t \in (1, d]$ .

The third result is on the transition asymptotics of the recurrence coefficients of the corresponding monic orthogonal polynomials; see (1.6).

*Theorem 3.* Let  $n \rightarrow \infty$  and  $t \rightarrow 1$ , then we have the asymptotic approximation of the recurrence coefficients

$$b_{n-1}^2 = \frac{1}{4} - 8 \left( \frac{q}{s} \right)' (t-1) + O\left(\frac{t-1}{n}\right) + O\left(\frac{\sqrt{t-1}}{n^2}\right) + O\left(\frac{1}{n^3}\right), \quad (1.12)$$

where  $q(s) = 4\sigma_{\text{JM}}(\frac{s^2}{16}) - \frac{s^2}{16} - (\alpha + \beta)^2 + \frac{1}{4}$ ,  $s = 4n \ln \varphi(t)$ ,  $\varphi(t) = t + \sqrt{t^2 - 1}$ , the derivative is taken with respect to  $s$  and the error term  $O\left(\frac{1}{n^3}\right)$  is uniform for  $t \in (1, d]$ .

*Remark 3.* The variable  $s = 4n \ln \varphi(t) \sim 4\sqrt{2} n \sqrt{t-1} \in (0, \infty)$  describes the gap between the hard edge and the algebraic singularity of the weight function (1.1). Theorems 1, 2 and 3 describe the transition of asymptotics of the Hankel determinants, the leading coefficients and the recurrence coefficients associated with weights having two different hard edge singularities: On the one side it is of the form  $(1 - x^2)^{\alpha+\beta}$  as  $s \rightarrow 0^+$ , on the other side with  $(1 - x^2)^\beta$  as  $s \rightarrow \infty$ . For fixed  $s$  taken in the transition region  $(0, \infty)$ , we obtain the Painlevé type transition asymptotics in the double scaling limit. Moreover, as  $s \rightarrow 0$  and  $s \rightarrow \infty$ , the limiting asymptotics of these quantities agree with the known asymptotics for the modified Jacobi weight with fixed singularities.

Let  $s \rightarrow 0$  or  $s \rightarrow \infty$ , we obtain from Theorem 2 the asymptotics of the leading coefficients corresponding to Jacobi polynomials of different orders. We state the results in the following corollary, which are in consistence with those obtained in [18].

*Corollary 1.* (i) Let  $n \rightarrow \infty$  and  $t \rightarrow 1$  such that  $s = 4n \ln \varphi(t) \rightarrow 0^+$ , we have the asymptotic approximation of the leading coefficients of the orthonormal polynomials

$$\frac{\gamma_n}{2^n} = \frac{1}{\sqrt{\pi} D_1(\infty)} \left( 1 - \frac{4(\alpha + \beta)^2 - 1 + O(s^2)}{8n} + O\left(\frac{1}{n^2}\right) \right). \quad (1.13)$$

(ii) Let  $n \rightarrow \infty$  and  $t \rightarrow 1$  such that  $s \rightarrow \infty$ , we have the asymptotic approximation of the leading coefficients

$$\frac{\gamma_n}{2^n} = \frac{1}{\sqrt{\pi}D_t(\infty)} \left( 1 - \frac{4\beta^2 - 1}{8n} + o\left(\frac{1}{n}\right) \right). \quad (1.14)$$

Similarly, as  $s \rightarrow 0$  or  $s \rightarrow \infty$ , we extract from Theorem 3 the asymptotics of the recurrence coefficients corresponding to Jacobi polynomials with different parameters. The results are also consistent with those obtained in [18].

*Corollary 2.* (i) Let  $n \rightarrow \infty$  and  $t \rightarrow 1$  such that  $s \rightarrow 0^+$ , we have the following asymptotic approximation of the recurrence coefficients

$$b_{n-1}^2 = \frac{1}{4} - \frac{4(\alpha + \beta)^2 - 1 + O(s)}{16n^2} + O(n^{-3}). \quad (1.15)$$

(ii) Let  $n \rightarrow \infty$  and  $t \rightarrow 1$  such that  $s \rightarrow \infty$  and  $\frac{s^2}{n} \rightarrow 0$ , we have the asymptotic of the recurrence coefficients

$$b_{n-1}^2 = \frac{1}{4} - \frac{4\beta^2 - 1}{16n^2} + o\left(\frac{1}{n^2}\right). \quad (1.16)$$

To prove the main results, first we derive several differential identities for the leading coefficients and the logarithmic derivative of the Hankel determinants, relating to the solution of the matrix Riemann-Hilbert (RH) formulation for orthogonal polynomials. Then we make use of the results obtained in a preceding paper of the authors using the Deift-Zhou steepest descent method for the RH problems [8, 11, 12]. The derivation is given in [26] with full details, and is briefly reviewed in Section 4 below.

The rest of the paper is organized as follows. In Section 2, we formulate the model RH problem for the Painlevé III equation, which is introduced by the authors earlier in [26]. An equivalent  $\sigma$ -form of Painlevé III equation is then derived. In Section 3, we prove the differential identities for the Hankel determinants and the leading coefficients in terms of the RH problem for the orthogonal polynomials associated with the weight (1.1). The identities are the starting points of our analysis in later sections. In Section 4, we outline the notations and formulas resulted from the RH analysis, obtained previously by the authors in [26]. The proofs of Theorems 1, 2, 3 are provided in the last section, Section 5.

## 2 Painlevé III equation and $\sigma$ -form of Painlevé III equation

In [26], to construct the local parametrix in the nonlinear steepest descent analysis of the RH problems, Xu and Zhao introduce a modified version of the Painlevé V equation which is equivalent to the Painlevé III equation after a Möbius transformation.

Proposition 1. (Xu and Zhao [26]) Assume that  $y(s)$  solves

$$\frac{d^2 y}{ds^2} - \frac{2y}{y^2 - 1} \left( \frac{dy}{ds} \right)^2 + \frac{1}{s} \frac{dy}{ds} + \frac{y(y^2 + 1)}{4(y^2 - 1)} + \frac{y}{2s} + \alpha \frac{y}{s} + \left( \beta - \frac{1}{2} \right) \frac{y^2 + 1}{2s} = 0, \quad (2.1)$$

where  $\alpha$  and  $\beta$  are constants. The equation is converted to a generalized Painlevé V equation by putting  $\omega = y^2$ , so that

$$\frac{d^2 \omega}{ds^2} - \left( \frac{1}{\omega - 1} + \frac{1}{2\omega} \right) \left( \frac{d\omega}{ds} \right)^2 + \frac{1}{s} \frac{d\omega}{ds} + \frac{(2\alpha + 1)\omega}{s} + \frac{\omega(\omega + 1)}{2(\omega - 1)} \pm \left( \beta - \frac{1}{2} \right) \frac{\sqrt{\omega}}{s} (\omega + 1) = 0, \quad (2.2)$$

which is reduced to the classical Painlevé V equation for  $\beta = \frac{1}{2}$ . Applying the Möbius transformation  $v(s) = \frac{y(s)+1}{y(s)-1}$  turns the equation (2.1) into the Painlevé III equation

$$\frac{d^2 v}{ds^2} - \frac{1}{v} \left( \frac{dv}{ds} \right)^2 + \frac{1}{s} \frac{dv}{ds} + \frac{1}{s} \left( \frac{\alpha - \beta}{2} v^2 + \frac{\alpha - \beta + 1}{2} \right) - \frac{v^3}{16} + \frac{1}{16v} = 0. \quad (2.3)$$

Moreover, the equation (2.1) is the compatibility condition for the Lax pair

$$\Psi_\lambda(\lambda, s) = \left( \frac{s\sigma_3}{2} + \frac{A(s)}{\lambda - \frac{1}{2}} + \frac{B(s)}{\lambda + \frac{1}{2}} + \frac{(\beta - \frac{1}{2})\sigma_1}{\lambda} \right) \Psi(\lambda, s), \quad (2.4)$$

$$\Psi_s(\lambda, s) = \left( \frac{\lambda\sigma_3}{2} + u(s)\sigma_1 \right) \Psi(\lambda, s), \quad (2.5)$$

where  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  are the Pauli matrices,

$$A(s) = \sigma_1 B(s) \sigma_1, \quad \text{and} \quad B(s) = \begin{pmatrix} b(s) - \frac{\alpha}{2} & -(b(s) - \alpha)y(s) \\ b(s)/y(s) & -b(s) + \frac{\alpha}{2} \end{pmatrix}, \quad (2.6)$$

with  $y(s)$  being a particular solution of (2.1), while  $b(s)$  and  $u(s)$  are determined by the equations

$$s \frac{dy}{ds} = -\frac{sy}{2} + \frac{b(y^2 - 1)^2}{y} - \alpha(y^2 - 1)y - \left( \beta - \frac{1}{2} \right) (y^2 - 1) \quad (2.7)$$

and

$$u(s) = \frac{b(s)/y(s) - (b(s) - \alpha)y(s)}{s} + \frac{\beta - \frac{1}{2}}{s}. \quad (2.8)$$

□

In view of the symmetry  $\sigma_1 \Psi(-\lambda) \sigma_1 = \Psi(\lambda)$ , new Lax pair of differential equations are obtained by applying the transformation

$$\Psi_0(\zeta, s) = e^{-\frac{\pi i}{4} \sigma_3} \zeta^{\frac{1}{4} \sigma_3} \frac{I + i \sigma_2}{\sqrt{2}} \Psi \left( \sqrt{\zeta}, s \right) e^{\frac{\pi i}{4} \sigma_3}, \quad \arg \zeta \in (-\pi, \pi). \quad (2.9)$$



Then a model RH problem for  $\Psi_0(\zeta, s)$  in the  $\zeta$ -plane is formulated, and its unique solvability is proved for  $s > 0$ ; see [26]. For later use, we recall the asymptotic behavior of the model RH problem for  $\Psi_0(\zeta, s)$  as follows:

$$\Psi_0(\zeta) = \zeta^{\frac{1}{4}\sigma_3} \frac{I - i\sigma_1}{\sqrt{2}} \left( I + \frac{\frac{\sigma(s)}{s}\sigma_3 + iu(s)\sigma_1}{\sqrt{\zeta}} + O\left(\frac{1}{\zeta}\right) \right) e^{\frac{s\sqrt{\zeta}}{2}\sigma_3} \quad (2.10)$$

as  $\zeta \rightarrow \infty$  for  $\arg \zeta \in (-\pi, \pi)$ , where  $s \in (0, \infty)$ , the function  $u$  is defined in (2.8) and  $\sigma$  is defined as

$$\sigma(s) = (b(s) - \alpha/2)s - (su)^2; \quad (2.11)$$

also,

$$\Psi_0(\zeta) = E_0 \left( I + \sum_{k=1}^{\infty} E_k (\zeta - 1/4)^k \right) (\zeta - 1/4)^{\frac{1}{2}\alpha\sigma_3} \quad (2.12)$$

as  $\zeta \rightarrow 1/4$  for  $\arg(\zeta - 1/4) \in (-\pi, \pi)$ , where the coefficients can be determined by substituting (2.12) into (2.4) and (2.9). For example, the leading coefficient is

$$E_0 = \sqrt{\frac{b-\alpha}{2\alpha}} 2^{-\frac{1}{2}\sigma_3} \begin{pmatrix} 1 + y(s) & \frac{ib(s)}{y(s)(b(s)-\alpha)} + i \\ i(y(s) - 1) & \frac{b(s)}{y(s)(b(s)-\alpha)} - 1 \end{pmatrix}, \quad (2.13)$$

and the  $(1, 1)$  entry of  $E_1$  can be represented as

$$(E_1)_{11} = -\frac{1}{\alpha} \left( \sigma - su + \alpha^2 + \beta^2 - \frac{1}{4} \right). \quad (2.14)$$

It is also noted that (see [26, Sec. 2.1], with  $\Theta = -\alpha$ )

$$\sigma' = b - \frac{\alpha}{2}, \quad (su)' = \frac{1}{2} \left( \frac{b}{y} + y(b - \alpha) \right). \quad (2.15)$$

## 2.1 Jimbo-Miwa-Okamoto $\sigma$ -form of the Painlevé III equation

We derive the Jimbo-Miwa-Okamoto  $\sigma$ -form of the Painlevé III equation.

Let

$$q(s) = \sigma(s) - us, \quad (2.16)$$

where  $\sigma(s)$  and  $u(s)$  appear in the coefficients of the asymptotic behavior of  $\Psi_0$  at infinity, given in (2.10), then by the relation

$$s\sigma' - \sigma = (su)^2; \quad (2.17)$$

cf. (2.11) and (2.15), we have

$$\left( \frac{q}{s} \right)' = u^2 - u', \quad (2.18)$$

and

$$sq' = \sigma + (su)^2 - s(su)'. \quad (2.19)$$

Then taking derivative on both sides of (2.19), using the definition of  $q$  and the equation

$$\frac{d^2(su)}{ds^2} = 2u\sigma' + \frac{1}{4} \left( su - \beta + \frac{1}{2} \right);$$

see [26, Sec. 2.1] with  $\Theta = -\alpha$  and  $\gamma = \beta - \frac{1}{2}$ , and in view of (2.17), we obtain

$$u = \frac{\frac{\beta}{4} - \frac{1}{8} - q''}{2q' + \frac{s}{4}}. \quad (2.20)$$

Substituting (2.20) into (2.18), we get the following third order equation

$$\frac{q'''}{q' + \frac{s}{8}} - \frac{(q'' - \frac{\beta}{4} + \frac{1}{8})(q'' + \frac{\beta}{4} + \frac{1}{8})}{2(q' + \frac{s}{8})^2} - 2\left(\frac{q}{s}\right)' = 0. \quad (2.21)$$

By the transformation

$$q(s) = 4\sigma_{\text{JM}}(s^2/16) - s^2/16 - c_2 + \frac{1}{4}, \quad (2.22)$$

we get from (2.21) the new third order equation

$$III := 2s^2\sigma'_{\text{JM}}\sigma''_{\text{JM}} - s^2\sigma''_{\text{JM}}{}^2 + 2s\sigma'_{\text{JM}}\sigma''_{\text{JM}} - 8s\sigma'_{\text{JM}}{}^3 + (4\sigma_{\text{JM}} + s - c_2)\sigma'_{\text{JM}}{}^2 + c_0 = 0, \quad (2.23)$$

where  $c_2 = (\alpha + \beta)^2$ ,  $c_0 = \frac{\beta^2}{16}$ .

Let  $II$  denote the left-hand side of Jimbo-Miwa-Okamoto  $\sigma$ -form (1.7) of the Painlevé III equation, namely,

$$II(s) := (s\sigma''_{\text{JM}})^2 + \sigma'_{\text{JM}}(\sigma_{\text{JM}} - s\sigma'_{\text{JM}})(4\sigma'_{\text{JM}} - 1) - c_2\sigma_{\text{JM}}'^2 - c_1\sigma'_{\text{JM}} - c_0, \quad (2.24)$$

where  $c_0$  and  $c_2$  are defined in (2.23), and  $c_1$  is a certain constant to be determined, then

$$\left(\frac{II}{\sigma_{\text{JM}}'^2}\right)' \frac{\sigma_{\text{JM}}'^3}{\sigma_{\text{JM}}''} + II = III. \quad (2.25)$$

From the above equality we see that the third order nonlinear equation (2.23) is equivalent to the Jimbo-Miwa-Okamoto  $\sigma$ -form (1.7) of the Painlevé III equation. Indeed, if  $\sigma_{\text{JM}}$  satisfies the  $\sigma$ -form (1.7) for arbitrary coefficient  $c_1$ , in view of (2.24) we have  $II(s) = c\sigma'_{\text{JM}}$  for a constant  $c$ . Substituting it into (2.25) then yields (2.23). Conversely, if  $\sigma_{\text{JM}}$  solves (2.23), then (2.25) is reduced to

$$\frac{II'}{II} = \frac{\sigma_{\text{JM}}''}{\sigma_{\text{JM}}'}.$$

Solving this equation gives

$$II(s) = c\sigma'_{\text{JM}},$$

where  $c$  is a constant. Hence  $\sigma_{\text{JM}}$  satisfies the Jimbo-Miwa-Okamoto  $\sigma$ -form (1.7) of the Painlevé III equation. The coefficient  $c_1 = -\frac{1}{2}\beta(\alpha + \beta)$ , as can be determined by substituting the behavior of  $\sigma_{\text{JM}}$  at infinity into (1.7); see (2.30).

We summarize the above derivation as follows:

*Proposition 2. Let*

$$q(s) = \sigma(s) - su(s),$$

*where  $\sigma(s)$  and  $u(s)$  appear in (2.10) describing the asymptotic behavior of  $\Psi_0$  at infinity, then  $q(s)$  satisfies the third order nonlinear differential equation*

$$\frac{q'''}{q' + \frac{s}{8}} - \frac{(q'' - \frac{\beta}{4} + \frac{1}{8})(q'' + \frac{\beta}{4} + \frac{1}{8})}{2(q' + \frac{s}{8})^2} - 2\left(\frac{q}{s}\right)' = 0.$$

*By the transformation*

$$q(s) = 4\sigma_{\text{JM}}(s^2/16) - s^2/16 - c_2 + \frac{1}{4},$$

*the above third order equation is turned into the Jimbo-Miwa-Okamoto  $\sigma$ -form of the Painlevé III equation*

$$(s\sigma''_{\text{JM}})^2 + \sigma'_{\text{JM}}(\sigma_{\text{JM}} - s\sigma'_{\text{JM}})(4\sigma'_{\text{JM}} - 1) - c_2\sigma_{\text{JM}}'^2 - c_1\sigma'_{\text{JM}} - c_0 = 0, \quad (2.26)$$

*where  $c_2 = (\alpha + \beta)^2$ ,  $c_1 = -\frac{1}{2}\beta(\alpha + \beta)$  and  $c_0 = \frac{\beta^2}{16}$ .  $\square$*

Noting that for  $\beta = 0$ , the equation (2.26) is reduced to the special Jimbo-Miwa-Okamoto  $\sigma$ -form (1.4) of the Painlevé III equation, as appeared in [22].

In [26, Prop. 2], it is proved that the RH problem for  $\Psi_0(\zeta, s)$  has a unique solution for  $s \in (0, \infty)$ . The nonlinear steepest descent analysis of the RH problem for  $\Psi_0(\zeta, s)$  is also carried out as  $s \rightarrow 0$  and  $s \rightarrow \infty$ . As a by-product, the asymptotics of the specific Painlevé function are then obtained. We collect the results in the proposition that follows, obtaining directly from Proposition 3 in [26].

*Proposition 3. The functions  $\sigma(s)$  and  $u(s)$  are analytic in  $s \in (0, \infty)$ . For these and several other auxiliary functions, we have the asymptotic behavior as  $s \rightarrow \infty$ :*

$$\begin{aligned} y(s) &= \pm \frac{2\beta - 1}{s} + O\left(\frac{1}{s^2}\right), \\ \sigma(s) &= -\frac{\alpha}{2}s - \left(\beta - \frac{1}{2}\right)^2 - \frac{4\alpha(\beta - \frac{1}{2})^2}{s} + O\left(\frac{1}{s^2}\right), \\ b(s) &= \frac{4\alpha(\beta - \frac{1}{2})^2}{s^2} + O\left(\frac{1}{s^3}\right), \\ u(s) &= \frac{\beta - \frac{1}{2}}{s} + \frac{4\alpha(\beta - \frac{1}{2})}{s^2} + \frac{16\alpha^2(\beta - \frac{1}{2})}{s^3} + O\left(\frac{1}{s^4}\right). \end{aligned} \quad (2.27)$$

As  $s \rightarrow 0$ , they behave as

$$\begin{aligned}
y(s) &= 1 + \frac{\beta - \alpha}{2(\alpha + \beta)}s + O(s^2) + O(s^{2+2(\alpha+\beta)}), \\
\sigma(s) &= -(\alpha + \beta)^2 - \frac{1}{4} + \frac{\alpha}{\alpha + \beta} \left( -\frac{s^2}{32} + 2C_0(\alpha, \beta) \left( \frac{s^2}{16} \right)^{1+\alpha+\beta} \right) (1 + O(s^2)), \\
b(s) &= -\frac{(\alpha + \beta)^2}{s} + \frac{\alpha}{2} + \frac{3}{2} \frac{\alpha}{\alpha + \beta} \left( -\frac{s}{16} + 4C_0(\alpha, \beta) \left( \frac{s^2}{16} \right)^{\frac{1}{2}+\alpha+\beta} \right) (1 + O(s^2)), \\
su(s) &= -\frac{1}{2} - \frac{\alpha}{\alpha + \beta} \left( -\frac{s^2}{32} + 2C_0(\alpha, \beta) \left( \frac{s^2}{16} \right)^{1+\alpha+\beta} \right) (1 + O(s^2)),
\end{aligned} \tag{2.28}$$

where

$$C_0(\alpha, \beta) = \frac{1}{2^{2+2(\alpha+\beta)}} \frac{\Gamma(1 - \alpha - \beta)\Gamma(\beta + 1)}{\Gamma(1 - \alpha)\Gamma(\alpha + \beta + 2)\Gamma(\alpha + \beta + 1)}. \quad \square$$

As a corollary of Proposition 3, we have the following asymptotic behavior of  $\sigma_{\text{JM}}$ .

*Corollary 3.* The Jimbo-Miwa-Okamoto  $\sigma$ -function  $\sigma_{\text{JM}}(s)$  is analytic for  $s \in (0, \infty)$  and satisfies the boundary conditions

$$\sigma_{\text{JM}}(s) = \frac{\beta}{4(\alpha + \beta)}s + C(\alpha, \beta) \left( \frac{s}{4} \right)^{1+\alpha+\beta} + O(s^{2+\alpha+\beta}) + O(s^2) \quad \text{as } s \rightarrow 0^+, \tag{2.29}$$

and

$$\sigma_{\text{JM}}(s) = \frac{1}{4}s - \frac{\alpha}{2}\sqrt{s} + \frac{1}{4}(\alpha^2 + 2\alpha\beta) - \frac{\alpha(\beta^2 - \frac{1}{4})}{4\sqrt{s}} + O(s^{-1}) \quad \text{as } s \rightarrow +\infty, \tag{2.30}$$

with

$$C(\alpha, \beta) = \frac{\alpha\Gamma(1 - \alpha - \beta)\Gamma(\beta + 1)}{(\alpha + \beta)\Gamma(1 - \alpha)\Gamma(\alpha + \beta + 2)\Gamma(\alpha + \beta + 1)}.$$

The asymptotic behavior of the  $\sigma$ -function  $\sigma_{\text{JM}}(s)$  has been considered in [15] and [17]. To compare the results with those obtained in [17], we take  $\zeta(s) = -\sigma_{\text{JM}}(4s) + s$ , then by (2.26) we have

$$(s\zeta'')^2 = 4\zeta'(\zeta' - 1)(\zeta - s\zeta') + ((\alpha + \beta)\zeta' - \alpha)^2. \tag{2.31}$$

The related  $\tau$ -function is defined as

$$\zeta(s) = s \frac{d}{ds} \ln \tau(s) - \alpha\beta + s$$

in [17] in our notations. Then by (2.29), we get the asymptotic of the  $\tau$ -function

$$\tau(s) = cs^{\alpha\beta} \left[ 1 - \frac{\beta s}{\alpha + \beta} - \frac{C(\alpha, \beta)}{1 + \alpha + \beta} s^{1+\alpha+\beta} + O(s^{2+\alpha+\beta}) + O(s^2) + O(s^{2(1+\alpha+\beta)}) \right] \tag{2.32}$$

with an arbitrary constant  $c$ . The result is in consistence with [17, Thm. 3.2]. It is noted that, in [17], more general  $\tau$ -function with three complex parameters is considered, thus more restrictions on the parameters are needed. Yet in [17], in our notation, a restriction  $-1 < \text{Re}(\alpha + \beta) \leq 0$  is brought in, and the error estimate therein is given as  $O(s^{2(1+\text{Re}(\alpha+\beta))})$ ; see [17, Thm. 3.2].

### 3 Riemann-Hilbert problem for orthogonal polynomials and differential identities

Let  $\pi_n(z)$  be the monic polynomial of degree  $n$  with respect to the weight  $w(x) = w(x; t)$  in (1.1), then

$$Y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i} \int_{-1}^1 \frac{\pi_n(s)w(s)}{s-z} ds \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(z) & -\gamma_{n-1}^2 \int_{-1}^1 \frac{\pi_{n-1}(s)w(s)}{s-z} ds \end{pmatrix}, \quad (3.1)$$

is the unique matrix-valued function analytic in  $\mathbb{C} \setminus [-1, 1]$ , fulfilling the jump condition

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \quad \text{for } x \in (-1, 1), \quad (3.2)$$

the asymptotic condition at infinity

$$Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad \text{as } z \rightarrow \infty, \quad (3.3)$$

and certain behavior demonstrating weak singularities at  $z = \pm 1$ ; see [13] and [26].

To derive the asymptotic behavior of the Hankel determinants, the leading coefficients and the recurrence coefficients, we establish differential identities to represent several quantities in terms of the matrix-valued function  $Y$  in (3.1).

*Lemma 1. Let*

$$h_n = \gamma_n^{-2} = \int_{-1}^1 \pi_n^2(x) w(x) dx, \quad (3.4)$$

*then  $h_n$  can be expressed in terms of  $Y(\pm t)$  as*

$$\frac{d}{dt} h_n = 2\pi i \alpha (Y_{11}(-t)Y_{12}(-t) - Y_{11}(t)Y_{12}(t)), \quad (3.5)$$

*where  $\gamma_n$  is the leading coefficient of the orthonormal polynomial of degree  $n$ , and  $Y_{ij}(z)$  denotes the  $(i, j)$  entry of  $Y(z)$  given in (3.1).*

**PROOF.** Taking derivative with respect to  $t$  on both sides of (3.4) and making use of the orthogonality, we arrive at

$$\frac{d}{dt} h_n = \int_{-1}^1 \pi_n^2(x) \frac{\partial}{\partial t} w(x) dx.$$

It follows from (1.1) that

$$\frac{\partial}{\partial t} w(x; t) = \alpha w(x; t) \left( \frac{1}{t-x} + \frac{1}{t+x} \right).$$

Then the lemma is obtained by partial fraction decomposition of  $\pi_n(x)/(x \pm t)$  and again using the orthogonality.  $\square$

For the Hankel determinants, we also have

*Lemma 2. Let*

$$H_n = \frac{d}{dt} \ln D_n(t), \quad (3.6)$$

*then the following differential identity holds,*

$$H_n = \alpha \left( (Y^{-1}Y'_z)_{11}(t) - (Y^{-1}Y'_z)_{11}(-t) \right). \quad (3.7)$$

PROOF. By the well-known relation between the Hankel determinants and the leading coefficients

$$\gamma_k^2 = D_k/D_{k+1}, \quad D_n = \gamma_{n-1}^{-2} \gamma_{n-2}^{-2} \cdots \gamma_0^{-2}; \quad (3.8)$$

cf. Szegő [21, (2.2.15)], and the orthogonal relation

$$\int_{-1}^1 p_n(x)^2 w(x) dx = 1, \quad (3.9)$$

where  $p_n(z) = \gamma_n \pi_n(z)$  is the orthonormal polynomial with respect to (1.1), we get

$$H_n = \int_{-1}^1 \sum_{k=0}^{n-1} \gamma_k^2 \pi_k^2(x) \frac{\partial w(x)}{\partial t} dx = \alpha \int_{-1}^1 \sum_{k=0}^{n-1} \gamma_k^2 \pi_k^2(x) w(x) \left( \frac{1}{t-x} + \frac{1}{t+x} \right) dx. \quad (3.10)$$

Now the Christoffel-Darboux formula ([21, (3.2.4)]) applies and we have

$$\sum_{k=1}^{n-1} p_k^2(x) = \gamma_{n-1}^2 \left( \pi_{n-1} \frac{d}{dx} \pi_n - \pi_n \frac{d}{dx} \pi_{n-1} \right) \quad (3.11)$$

Substituting (3.11) into (3.10) and using the fraction decomposition techniques and the orthogonal relation, we obtain (3.7).  $\square$

## 4 Nonlinear steepest descent analysis

The nonlinear steepest descent analysis for the orthogonal polynomials has been provided by two of the present authors in [26, Sec. 3]. The central piece is the construction of the local parametrix in a domain containing singularities  $z = 1$  and  $z = t$ , in which a modified Painlevé V equation is involved. It is shown that the equation is equivalent to the Painlevé III equation after a Möbius transformation. In this section, we briefly review the results and collect several formulas to be used in the investigation of the Hankel determinants and the recurrence coefficients.

In [26, Sec. 3.4], applying a certain normalization at infinity,  $Y(z)$  in (3.1) is approximated by

$$N_t(z) = D_t(\infty)^{\sigma_3} M_1^{-1} a(z)^{-\sigma_3} M_1 D_t(z)^{-\sigma_3} \quad (4.1)$$

for  $z$  kept away from  $[-1, 1]$ , where  $M_1 = \frac{1}{\sqrt{2}}(I + i\sigma_1)$ ,  $a(z) = \left(\frac{z-1}{z+1}\right)^{1/4}$  for  $z \in \mathbb{C} \setminus [-1, 1]$  with branches chosen such that  $\arg(z \pm 1) \in (-\pi, \pi)$  and thus  $a(x)$  is positive for  $x > 1$  and  $a_+(x)/a_-(x) = i$  for  $x \in (-1, 1)$ , and the Szegő function associated with  $w(x)$  takes the form

$$D_t(z) = \left(\frac{z^2 - 1}{\varphi(z)^2}\right)^{\beta/2} \exp\left(\frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^1 \frac{\ln\{(t^2 - x^2)^\alpha h(x)\}}{\sqrt{1 - x^2}} \frac{dx}{z - x}\right), \quad z \in \mathbb{C} \setminus [-1, 1], \quad (4.2)$$

in which  $\varphi(z) = z + \sqrt{z^2 - 1}$  is analytic in  $\mathbb{C} \setminus [-1, 1]$  and  $\varphi(z) \approx 2z$  as  $z \rightarrow \infty$ . In (4.1),

$$D_t(\infty) = \lim_{z \rightarrow \infty} D_t(z) = 2^{-\beta} \exp\left(\frac{1}{2\pi} \int_{-1}^1 \frac{\ln[(t^2 - x^2)^\alpha h(x)] dx}{\sqrt{1 - x^2}}\right). \quad (4.3)$$

In a disc  $U(1, \delta)$  centered at  $z = 1$  with fixed positive small radius  $\delta$ , containing the hard edge  $z = 1$  and the algebraic singularity  $z = t$  of the weight function, the local parametrix is constructed as

$$P^{(1)}(z) = E(z) \Psi_0(f_t(z), 2n\sqrt{\rho_t}) \varphi(z)^{-n\sigma_3} W(z)^{-\frac{1}{2}\sigma_3}, \quad (4.4)$$

where  $W(z) = (z^2 - 1)^\beta (z^2 - t^2)^\alpha h(z)$ ,  $\arg(z \pm 1) \in (-\pi, \pi)$ ,  $\arg(z \pm t) \in (-\pi, \pi)$ ,  $\Psi_0(\zeta) = \Psi_0(\zeta, s)$  is the solution to the model RH problem related to the Painlevé III equations; see (2.9). Also, in (4.4),

$$E(z) = N_t(z) W(z)^{\frac{1}{2}\sigma_3} \{G(f_t(z))\}^{-1}, \quad (4.5)$$

where  $G(\zeta)$  is a specific matrix function defined as

$$G(\zeta) = \zeta^{\frac{1}{4}\sigma_3} \frac{I - i\sigma_1}{\sqrt{2}} \exp\left\{\left(\frac{\alpha\sqrt{\zeta}}{2} \int_0^{\frac{1}{4}} \frac{1}{\sqrt{\tau}} \frac{d\tau}{\tau - \zeta}\right) \sigma_3\right\}, \quad \zeta \in \mathbb{C} \setminus (-\infty, 1/4], \quad (4.6)$$

and the conformal mapping

$$f_t(z) = \frac{(\ln \varphi(z))^2}{\rho_t} = \frac{2(z - 1)}{\rho_t} (1 + O(z - 1)), \quad z \in U(1, \delta), \quad (4.7)$$

with  $\rho_t = 4(\ln \varphi(t))^2$  such that  $\rho_t = 8(t - 1) + O((t - 1)^2)$  as  $t \rightarrow 1$ .

Accordingly,  $Y(z)$  in (3.1) is approximated by the local parametrix in  $U(1, \delta)$ . More precisely, we have

$$Y(z) = 2^{-n\sigma_3} R(z) E(z) \Psi_0(f_t(z)) W(z)^{-\frac{1}{2}\sigma_3}, \quad t < z < 1 + \delta, \quad (4.8)$$

where  $E$  and  $f_t$  is defined in (4.5) and (4.7), respectively, and

$$R(z) = I + O(n^{-1}), \quad (4.9)$$

uniformly for  $z$  in the whole complex plane.

## 5 Proof of the theorems

In the present section, we apply the differential identities in Lemma 1 and Lemma 2 and the results obtained via the nonlinear steepest descent analysis for the orthogonal polynomials summarized in Section 4, to derive the asymptotics of the Hankel determinants, the leading coefficients and the recurrence coefficients.

Substituting (4.1) into (4.5), we have

$$E(z) = D_t^{\sigma_3}(\infty) M_1^{-1} a(z)^{-\sigma_3} M_1 D_t(z)^{-\sigma_3} (W(z))^{\frac{1}{2}\sigma_3} G^{-1}(f_t(z)), \quad (5.1)$$

where  $a(z) = \left(\frac{z-1}{z+1}\right)^{\frac{1}{4}}$  with  $\arg(z \pm 1) \in (-\pi, \pi)$ , the Szegő function  $D_t(z)$ , and the auxiliary functions  $G$  and  $f_t$  are defined in (4.2), (4.6) and (4.7), respectively.

Making a change of variables  $s = \sqrt{\tau}$  in the integral in (4.6), we get a simpler representation for  $G$ , namely,

$$G(\zeta) = \zeta^{\frac{1}{4}\sigma_3} \frac{I - i\sigma_1}{\sqrt{2}} \left( \frac{2\sqrt{\zeta} - 1}{2\sqrt{\zeta} + 1} \right)^{\frac{\alpha}{2}\sigma_3}. \quad (5.2)$$

By the Cauchy theorem, the integral in (4.2) for the Szegő function  $D_t(z)$  can be written as the summation of the following two integrals

$$\begin{aligned} \frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^1 \frac{\ln(t^2 - x^2)^\alpha}{\sqrt{1 - x^2}} \frac{dx}{z - x} &= \frac{\alpha}{2} \ln \frac{z^2 - t^2}{\varphi(z)^2} - \frac{\sqrt{z^2 - 1}}{2} \int_1^t \frac{\alpha}{\sqrt{x^2 - 1}} \frac{dx}{x - z} \\ &+ \frac{\sqrt{z^2 - 1}}{2} \int_{-t}^{-1} \frac{\alpha}{\sqrt{x^2 - 1}} \frac{dx}{x - z} \end{aligned} \quad (5.3)$$

and

$$\frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^1 \frac{\ln h(x)}{\sqrt{1 - x^2}} \frac{dx}{z - x} = \frac{1}{2} \ln h(z) - \frac{\sqrt{z^2 - 1}}{4\pi i} \int_\Gamma \frac{\ln h(x)}{\sqrt{x^2 - 1}} \frac{dx}{x - z}, \quad (5.4)$$

where  $\arg(z \pm 1) \in (-\pi, \pi)$  and  $\Gamma$  is an anti-clockwise loop in the analytic domain of  $h$  encircling  $[-1, 1]$ . Then the integrals on the right-hand side of (5.3) are expressed explicitly in terms of elementary functions as

$$\int_1^t \frac{1}{\sqrt{x^2 - 1}} \frac{dx}{x - z} = -\frac{1}{\sqrt{z^2 - 1}} \ln \left( \frac{zt + \sqrt{t^2 - 1}\sqrt{z^2 - 1} - 1}{z - t} \right)$$

and

$$\int_{-1}^{-t} \frac{1}{\sqrt{x^2 - 1}} \frac{dx}{x - z} = \frac{1}{\sqrt{z^2 - 1}} \ln \left( \frac{zt - \sqrt{t^2 + 1}\sqrt{z^2 - 1} - 1}{z + t} \right),$$

where  $\arg(z \pm 1) \in (-\pi, \pi)$  and the logarithmic functions take the principle branch. Thus we get an explicit expression of the Szegő function defined in (4.2)

$$\begin{aligned} \frac{D_t^2(z)}{W(z)} &= \varphi(z)^{-2(\alpha+\beta)} \exp \left( -\frac{\sqrt{z^2 - 1}}{2\pi i} \int_\Gamma \frac{\ln h(\zeta)}{\sqrt{\zeta^2 - 1}} \frac{d\zeta}{\zeta - z} \right) \\ &\times \left( \frac{z + t}{z - t} \frac{zt + \sqrt{z^2 - 1}\sqrt{t^2 - 1} - 1}{zt - \sqrt{z^2 - 1}\sqrt{t^2 - 1} + 1} \right)^\alpha, \end{aligned} \quad (5.5)$$



where in the fractional power we take the principle branches, and  $\arg(z \pm 1) \in (-\pi, \pi)$ . It is readily verified that  $\left(\frac{D_t^2(x)}{W(x)}\right)_+ \left(\frac{D_t^2(x)}{W(x)}\right)_- = 1$  for  $x \in (-1, 1)$ .

In view of (4.3) and using a residue calculation argument, we have

$$\frac{d}{dt} \ln D_t(\infty) = \frac{\alpha}{\sqrt{t^2 - 1}}, \quad (5.6)$$

from which we obtain

$$D_t(\infty) = D_1(\infty) \left(t + \sqrt{t^2 - 1}\right)^\alpha. \quad (5.7)$$

Now a combination of (5.1), (5.2) and (5.5) gives

$$E(z) = D_t^{\sigma_3}(\infty) M_1^{-1} a(z)^{-\sigma_3} M_1 \mathfrak{D}(z; t)^{\sigma_3} M_1 \zeta^{-\frac{1}{4}\sigma_3}, \quad (5.8)$$

where

$$\begin{aligned} \mathfrak{D}(z; t) = & \varphi(z)^{\alpha+\beta} \exp \left( \frac{\sqrt{z^2 - 1}}{4\pi i} \int_{\Gamma} \frac{\ln h(\zeta)}{\sqrt{\zeta^2 - 1}} \frac{d\zeta}{\zeta - z} \right) \\ & \times \left( \frac{(z+t)(\ln \varphi(z) - \ln \varphi(t))}{(z-t)(\ln \varphi(z) + \ln \varphi(t))} \right)^{-\frac{\alpha}{2}} \left( \frac{zt + \sqrt{z^2 - 1}\sqrt{t^2 - 1} - 1}{zt - \sqrt{z^2 - 1}\sqrt{t^2 - 1} + 1} \right)^{-\frac{\alpha}{2}}, \end{aligned} \quad (5.9)$$

and  $\zeta = f_t(z)$  is defined in (4.7).

Using Taylor expansions at  $z = t$ , we have

$$\left. \frac{\ln \varphi(z) - \ln \varphi(t)}{z - t} \right|_{z=t} = \frac{\varphi'(t)}{\varphi(t)} = \frac{1}{\sqrt{t^2 - 1}}.$$

Therefore, at  $z = t$  we have

$$\mathfrak{D}(t; t) = \varphi(t)^{\alpha+\beta} \exp \left( \frac{\sqrt{t^2 - 1}}{4\pi i} \int_{\Gamma} \frac{\ln h(\zeta)}{\sqrt{\zeta^2 - 1}} \frac{d\zeta}{\zeta - t} \right) \left( \frac{\ln \varphi(t)}{t\sqrt{t^2 - 1}} \right)^{\frac{\alpha}{2}}.$$

By using the behaviors of  $\varphi(t)$  and  $\ln \varphi(t)$  as  $t \rightarrow 1$ , and noting that

$$\frac{1}{\zeta - t} = \frac{1}{\zeta - 1} \sum_{n=0}^{\infty} \left( \frac{t-1}{\zeta-1} \right)^n \quad \text{for} \quad \left| \frac{t-1}{\zeta-1} \right| < 1,$$

we obtain

$$\mathfrak{D}(t; t) = 1 + \sum_{k=1}^{\infty} d_k (t-1)^{\frac{k}{2}} \quad (5.10)$$

for  $0 < t-1 < \delta$ , where  $\delta$  is a constant such that  $0 < \delta < 1$ , the coefficients  $d_k$  are explicitly computable and the first two are

$$d_1 = \frac{\sqrt{2}}{2} (2(\alpha + \beta) + e_0) \quad \text{and} \quad d_2 = \frac{1}{4} (2(\alpha + \beta) + e_0)^2 - \frac{2}{3} \alpha \quad (5.11)$$

with

$$e_0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{\ln h(\zeta)}{\sqrt{\zeta^2 - 1}} \frac{d\zeta}{\zeta - 1}. \quad (5.12)$$

For later use, we further compute the logarithmic derivative of  $\mathfrak{D}(z; t)$ . Repeatedly using Taylor expansions at  $z = t$ , we have

$$\left. \frac{d}{dz} \ln \left( \frac{\ln \varphi(z) - \ln \varphi(t)}{z - t} \right) \right|_{z=t} = \frac{1}{2} \left( \frac{\varphi''(t)}{\varphi'(t)} - \frac{\varphi'(t)}{\varphi(t)} \right) = -\frac{t}{2(t^2 - 1)}.$$

Similarly, we get

$$\left. \frac{d}{dz} \ln \left( \frac{\ln \varphi(z) + \ln \varphi(t)}{z + t} \right) \right|_{z=t} = \frac{1}{2\sqrt{t^2 - 1} \ln \varphi(t)} - \frac{1}{2t} = \frac{1}{2(t^2 - 1)} - \frac{5}{12} + O(t - 1),$$

and

$$\left. \frac{d}{dz} \ln \left( zt + \sqrt{z^2 - 1} \sqrt{t^2 - 1} - 1 \right) \right|_{z=t} = \frac{t}{t^2 - 1}.$$

Putting these formulas together, taking logarithm of (5.9) and differentiating both sides, and in view of (5.10), we have

$$\left. \frac{d}{dz} \ln \mathfrak{D}(z; t) \right|_{z=t} = \left( \alpha + \beta + \frac{e_0}{2} \right) \frac{1}{\sqrt{2(t-1)}} - \frac{\alpha}{3} + \sum_{k=1}^{\infty} d'_k (t-1)^{\frac{k}{2}}, \quad (5.13)$$

where  $e_0$  is defined in (5.12) and  $d'_k$  are computable constants.

Substituting (5.10) into (5.8), we get the expansion

$$E(t) = D_t^{\sigma_3}(\infty) M_1 \left( \sum_{k=0}^{\infty} C_k (t-1)^{k/2} \right) (2(t-1))^{\frac{1}{4}\sigma_3}, \quad (5.14)$$

where  $M_1 = \frac{1}{\sqrt{2}}(I + i\sigma_1)$ , the coefficients  $C_k$  are explicitly computable matrices and the first few are

$$C_0 = \begin{pmatrix} 1 & 0 \\ -\sqrt{2}id_1 & 1 \end{pmatrix} \quad \text{and} \quad C_1 = \begin{pmatrix} 0 & 0 \\ \frac{2\sqrt{2}}{3}i\alpha & 0 \end{pmatrix},$$

with  $d_1$  defined in (5.11). By (2.13), we get

$$E_0 = \sqrt{\frac{b-\alpha}{2\alpha}} \sqrt{2}^{-\sigma_3} \begin{pmatrix} 1 + y(s) & \frac{ib(s)}{y(s)(b(s)-\alpha)} + i \\ i(y(s)-1) & \frac{b(s)}{y(s)(b(s)-\alpha)} - 1 \end{pmatrix}, \quad (5.15)$$

where  $s = 4n \ln \varphi(t)$ . From the boundary conditions (2.27) and (2.28) for  $y(s)$  and  $b(s)$ , we can derive the asymptotic behavior

$$E_0 = \sqrt{-(b-\alpha)} E_{0,0} (I + O(s) + O(s^{2(\alpha+\beta+1)})) \quad \text{as } s \rightarrow 0^+, \quad (5.16)$$

and

$$E_0 = \sum_{k=0}^{\infty} E_{0,-k} s^{-k} \quad \text{as } s \rightarrow +\infty, \quad (5.17)$$

where  $E_{0,k}$  are computable constant matrices and

$$(E_{0,0})_{21} = 0. \quad (5.18)$$

## 5.1 Proof of Theorem 1

Now we substitute the asymptotics we obtained for  $Y$  into the differential identity (3.7) for  $\ln D_n(t)$ . First, we consider the case for  $z$  close to  $t$ . From (4.8), we obtain

$$Y^{-1}Y'_z = W(z)^{\frac{1}{2}\sigma_3}\Psi_0^{-1}(\zeta)E^{-1}(z)E'_z(z)\Psi_0(\zeta)W(z)^{-\frac{1}{2}\sigma_3} + W(z)^{\frac{1}{2}\sigma_3}\Psi_0^{-1}(\zeta)\Psi'_{0,z}(\zeta)W(z)^{-\frac{1}{2}\sigma_3} \\ - \frac{W'_z}{2W}\sigma_3 + W(z)^{\frac{1}{2}\sigma_3}\Psi_0^{-1}(\zeta)E^{-1}(z)R^{-1}(z)R'_z(z)E(z)\Psi_0(\zeta)W(z)^{-\frac{1}{2}\sigma_3},$$

where  $\zeta = f_t(z)$  is defined in (4.7). Noting that  $\zeta(t) = \zeta|_{z=t} = \frac{1}{4}$ . By using the behavior (2.12) of  $\Psi_0(\zeta)$  at  $\zeta = 1/4$ , and collecting (4.9), (5.14) and (5.16)-(5.18) together, eventually we have the  $(1, 1)$  entry of  $Y^{-1}Y'_z$  at  $z = t$ :

$$(Y^{-1}Y'_z)_{11}(t) = (\Psi_0^{-1}(\zeta(t))E^{-1}(t)E'_z(t)\Psi_0(\zeta(t)))_{11} + (\Psi_0^{-1}(\zeta)\Psi'_{0,z}(\zeta))_{11}\Big|_{z=t} \\ - \frac{1}{2}\frac{W'_z}{W}\Big|_{z=t} + O\left(\frac{s^l}{n}\right) + O\left(\frac{s}{n}\right), \quad (5.19)$$

where  $s = 4n \ln \varphi(t)$ ,  $l = \min(1, 2(\alpha + \beta + 1))$  and the error terms are uniform for  $t \in (1, d]$ .

Using (5.8), we further obtain

$$E^{-1}(t)E'_z(t) = \begin{pmatrix} I_1(t) & I_2(t) \\ I_3(t) & -I_1(t) \end{pmatrix}, \quad (5.20)$$

where

$$I_1(t) = \frac{a'(t)}{2a(t)} (\mathfrak{D}^2(t, t) + \mathfrak{D}^{-2}(t, t)) - \zeta'_z(t), \\ I_2(t) = \frac{i}{2} \left( \frac{a'(t)}{2a(t)} (\mathfrak{D}^2(t, t) - \mathfrak{D}^{-2}(t, t)) + \frac{d}{dz} \ln \mathfrak{D}(z, t) \Big|_{z=t} \right), \\ I_3(t) = 2i \left( \frac{a'(t)}{2a(t)} (\mathfrak{D}^2(t, t) - \mathfrak{D}^{-2}(t, t)) - \frac{d}{dz} \ln \mathfrak{D}(z, t) \Big|_{z=t} \right);$$

see (5.9) and (5.10) for the definition of  $\mathfrak{D}$ .

It is readily seen that

$$\frac{d \ln a(t)}{dt} = \frac{1}{2(t^2 - 1)}, \quad (5.21)$$

and

$$\frac{d\zeta}{dz}\Big|_{z=t} = \frac{1}{2\sqrt{t^2 - 1} \ln \varphi(t)}. \quad (5.22)$$

Then, from (5.10), (5.13), (5.21) and (5.22), we get the estimates as  $t \rightarrow 1^+$ , namely,

$$I_1(t) = \frac{1}{4} \left( (2(\alpha + \beta) + e_0)^2 - \frac{1}{3} \right) + O(t - 1), \quad (5.23)$$

$$I_2(t) = i \left( -\frac{1}{3}\alpha + \frac{2(\alpha + \beta) + e_0}{2\sqrt{2}\sqrt{t-1}} \right) + O(\sqrt{t-1}), \quad (5.24)$$

$$I_3(t) = O(\sqrt{t-1}). \quad (5.25)$$

From (2.12), we have

$$\left( \Psi_0^{-1}(\zeta(t)) E^{-1}(t) E'_z(t) \Psi_0(\zeta(t)) \right)_{11} = \left( E_0^{-1} \begin{pmatrix} I_1(t) & I_2(t) \\ I_3(t) & -I_1(t) \end{pmatrix} E_0 \right)_{11}, \quad (5.26)$$

where  $\zeta(t) = \frac{1}{4}$ , and  $E_0$  is defined in (2.13). Thus by (2.13) and (5.26), we obtain

$$\begin{aligned} \left( \Psi_0^{-1}(\zeta(t)) E^{-1}(t) E'_z(t) \Psi_0(\zeta(t)) \right)_{11} &= \frac{su - \beta + \frac{1}{2}}{\alpha} I_1(t) + \frac{2i(\sigma'(s) - (su)')}{\alpha} I_2(t) \\ &\quad - \frac{i(\sigma'(s) + (su)')}{2\alpha} I_3(t), \end{aligned} \quad (5.27)$$

where the derivative is taken with respect to  $s$ ,  $s = 4n \ln \varphi(t)$ .

By (2.12), we get

$$(\Psi_0^{-1}(\zeta) \Psi'_{0,z}(\zeta))_{11} = \zeta'_z(E_1)_{11} + \frac{\frac{1}{2}\alpha\zeta'_z}{\zeta - \frac{1}{4}}.$$

Recalling  $W(z) = (z^2 - 1)^\beta (z^2 - t^2)^\alpha h(z)$ ; cf. (4.4), we have

$$\frac{W'_z}{W} = \frac{2\beta z}{z^2 - 1} + \alpha \left( \frac{1}{z - t} + \frac{1}{z + t} \right) + \frac{1}{2} \frac{d}{dz} \ln h(z).$$

Expanding the left-hand side at  $z = t$ , we obtain

$$\left( \frac{\zeta'_z}{\zeta - \frac{1}{4}} - \frac{1}{z - t} \right) (t) = \frac{1}{2} \left( \frac{1}{\sqrt{t^2 - 1} \ln \varphi(t)} - \frac{t}{t^2 - 1} \right).$$

Collecting these formulas together, and using (2.14), we get

$$\begin{aligned} \left( (\Psi_0^{-1}(\zeta) \Psi'_{0,z}(\zeta))_{11} - \frac{1}{2} \frac{W'_z}{W} \right) \Big|_{z=t} &= - \frac{\sigma(s) - us + \beta^2 - \frac{1}{4} + \frac{\alpha^2}{2}}{2\alpha\sqrt{t^2 - 1} \ln \varphi(t)} - \frac{\alpha}{4} \frac{t}{t^2 - 1} \\ &\quad - \frac{\beta t}{t^2 - 1} - \frac{\alpha}{4t} - \frac{1}{2} \frac{d}{dz} \ln h(z) \Big|_{z=t}. \end{aligned} \quad (5.28)$$

Substituting (5.27) and (5.28) into (5.19), we obtain

$$\begin{aligned} (Y^{-1}Y'_z)_{11}(t) &= - \frac{\sigma(s) - us + \beta^2 - \frac{1}{4} + \frac{\alpha^2}{2}}{2\alpha\sqrt{t^2 - 1} \ln \varphi(t)} - \left( \frac{\alpha}{4} + \beta \right) \frac{t}{t^2 - 1} - \frac{\alpha}{4t} \\ &\quad - \frac{1}{2} \frac{d}{dz} \ln h(z) \Big|_{z=t} + \frac{su - \beta + \frac{1}{2}}{\alpha} I_1(t) + \frac{2i(\sigma'(s) - (su)')}{\alpha} I_2(t) \\ &\quad - \frac{i(\sigma'(s) + (su)')}{2\alpha} I_3(t) + O\left(\frac{s^l}{n}\right) + O\left(\frac{s}{n}\right). \end{aligned} \quad (5.29)$$

Applying a similar argument to the case  $z = -t$ , we obtain

$$\begin{aligned} (Y^{-1}Y'_z)_{11}(-t) &= \frac{\sigma(s) - us + \beta^2 - \frac{1}{4} + \frac{\alpha^2}{2}}{2\alpha\sqrt{t^2 - 1} \ln \varphi(t)} + \left( \frac{\alpha}{4} + \beta \right) \frac{t}{t^2 - 1} + \frac{\alpha}{4t} \\ &\quad - \frac{1}{2} \frac{d}{dz} \ln h(z) \Big|_{z=-t} + \frac{su - \beta + \frac{1}{2}}{\alpha} \tilde{I}_1(-t) + \frac{2i(\sigma'(s) - (su)')}{\alpha} \tilde{I}_2(-t) \\ &\quad - \frac{i(\sigma'(s) + (su)')}{2\alpha} \tilde{I}_3(-t) + O\left(\frac{s^l}{n}\right) + O\left(\frac{s}{n}\right) \end{aligned}$$

$$(5.30)$$

with

$$\tilde{I}_1(-t) = -\frac{1}{4} \left( (2(\alpha + \beta) + e_0)^2 - \frac{1}{3} \right) + O(t-1), \quad (5.31)$$

$$\tilde{I}_2(-t) = -i \left( -\frac{1}{3}\alpha + \frac{2(\alpha + \beta) + e_0}{2\sqrt{2}\sqrt{t-1}} \right) + O(\sqrt{t-1}), \quad (5.32)$$

$$\tilde{I}_3(-t) = O(\sqrt{t-1}). \quad (5.33)$$

Here use has been made of the fact that  $h(x)$  is an even function.

Substituting (5.29) and (5.30) into (3.7) yields

$$\begin{aligned} \frac{d}{dt} \ln D_n(t) = & \frac{1}{2} \left[ (2(\alpha + \beta) + e_0)^2 - \frac{1}{3} \right] \left( su - \beta + \frac{1}{2} \right) - \frac{\sigma(s) - us + \beta^2 - \frac{1}{4} + \frac{\alpha^2}{2}}{\sqrt{t^2 - 1} \ln \varphi(t)} \\ & - \left[ \frac{2(\alpha + \beta) + e_0}{\sqrt{2}\sqrt{t-1}} - \frac{2\alpha}{3} \right] (\sigma'(s) - (su)') - \frac{(\alpha^2 + 4\alpha\beta)t}{2(t^2 - 1)} \\ & - \frac{\alpha^2}{2t} - \frac{\alpha}{2} \left( \frac{d}{dz} \ln h(z) \Big|_{z=t} - \frac{d}{dz} \ln h(z) \Big|_{z=-t} \right) \\ & + O\left(\frac{s^l}{n}\right) + O\left(\frac{s}{n}\right) + O(\sqrt{t-1}), \end{aligned} \quad (5.34)$$

where  $()' = \frac{d}{ds}$ . Integrating both sides of this identity from  $1 + \varepsilon$  to some  $t > 1$  gives

$$\begin{aligned} \ln D_n(t) = & \ln D_n(1 + \varepsilon) - \int_{4n \ln \varphi(1+\varepsilon)}^{4n \ln \varphi(t)} \frac{\sigma(s) - us + \beta^2 - \frac{1}{4} + \frac{\alpha^2}{2}}{s} ds \\ & - \left( \frac{1}{2}\alpha^2 + 2\alpha\beta \right) \int_{1+\varepsilon}^t \frac{t}{t^2 - 1} dt - \frac{1}{2}\alpha^2 \ln t + \frac{1}{2}\alpha^2 \ln(1 + \varepsilon) \\ & - \frac{\alpha}{2} V(t) - \frac{\alpha}{2} V(-t) + \frac{\alpha}{2} V(1 + \varepsilon) + \frac{\alpha}{2} V(-1 - \varepsilon) + R_n(t) + o(1), \end{aligned} \quad (5.35)$$

holding uniformly for arbitrary  $\varepsilon > 0$ , where  $V(z) = \ln h(z)$ , and the remainder term

$$\begin{aligned} R_n(t) = & \frac{1}{2} \left( (2(\alpha + \beta) + e_0)^2 - \frac{1}{3} \right) \int_{1+\varepsilon}^t \left( su - \beta + \frac{1}{2} \right) dt \\ & - \int_{1+\varepsilon}^t \left( \frac{2(\alpha + \beta) + e_0}{\sqrt{2}\sqrt{t-1}} - \frac{2}{3}\alpha \right) (\sigma'(s) - (su)') dt. \end{aligned}$$

Since  $s = 4n \ln \varphi(t)$ , then for small  $t - 1$ , we may approximate the integral

$$\int_1^t \frac{t}{t^2 - 1} dt = \int_0^s \frac{1}{s} ds + O(t-1). \quad (5.36)$$

From (2.27) and (2.28), namely the boundary conditions for  $u$  and  $\sigma$ , we get the estimates

for the integrals

$$\begin{aligned}\int_1^t \left( su - \beta + \frac{1}{2} \right) dt &= O(t-1), \\ \int_1^t (\sigma - su)' dt &= o(1), \\ \int_1^t \frac{1}{\sqrt{t^2-1}} (\sigma - su)' dt &= o(1),\end{aligned}$$

which give the estimate for  $R_n(t)$  as  $\varepsilon \rightarrow 0^+$

$$|R_n(t)| = O(\sqrt{t-1}) = o(1). \quad (5.37)$$

Letting  $\varepsilon \rightarrow 0^+$ , substituting (5.36) and (5.37) into (5.35) and making use of the fact that  $h(x)$  is an even function, we obtain

$$\begin{aligned}\ln D_n(t) &= \ln D_n(1) - \alpha V(t) + \alpha V(1) - \frac{1}{2} \alpha^2 \ln t \\ &\quad - \int_0^s \frac{\sigma(s) - us + (\alpha + \beta)^2 - \frac{1}{4}}{s} ds + o(1),\end{aligned} \quad (5.38)$$

where  $s = 4n \ln \varphi(t)$ ,  $\varphi(t) = t + \sqrt{t^2-1}$ . The convergence of the integral is guaranteed by the initial condition of  $\sigma(s)$  and  $u(s)$  in (2.28).

The asymptotic approximation for  $\ln D_n(1)$  has been given in [9, Thm. 1.20] as

$$\begin{aligned}\ln D_n(1) &= - (n^2 + 2n(\alpha + \beta) + 1) \ln 2 + \left( \alpha + \beta)^2 - \frac{1}{4} \right) \ln \frac{n}{4} \\ &\quad + \left( n + \alpha + \beta + \frac{1}{2} \right) \ln 2\pi + 2 \ln \frac{G(\frac{1}{2})}{G(\alpha + \beta + 1)} \\ &\quad + (n + \alpha + \beta) V_0 - (\alpha + \beta) V(1) + \frac{1}{2} \sum_{k=1}^{\infty} k V_k^2,\end{aligned} \quad (5.39)$$

where  $V_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ki\theta} \ln h(\cos(\theta)) d\theta$ ,  $k = 0, 1, \dots$ . And the Barnes  $G$ -function is defined by the product

$$G(1+z) = (2\pi)^{\frac{z}{2}} e^{-\frac{z+z^2(1+\gamma_E)}{2}} \prod_{k=1}^{\infty} \left( \left( 1 + \frac{z}{k} \right)^k e^{\frac{z^2}{2k} - z} \right), \quad (5.40)$$

where  $\gamma_E$  is the Euler constant. The Barnes  $G$ -function satisfies the well-known recurrence relation

$$G(z+1) = G(z)\Gamma(z),$$

where  $G(1) = 1$ , and  $\Gamma(z)$  is the gamma function.

Substituting (5.39) into (5.38) yields (1.8). Thus completing the proof of Theorem 1.

## 5.2 Proof of Theorem 2

From (2.12), (4.8) and (4.9), we obtain

$$Y(t) = 2^{-n\sigma_3}(I + O(1/n))E(t)E_0\{l(t)\}^{\sigma_3}, \quad (5.41)$$

where  $l(t) = \lim_{z \rightarrow t}(f_t(z) - \frac{1}{4})^{\frac{\alpha}{2}}W(z)^{-\frac{1}{2}}$ ,  $E_0$  and  $E(t)$  are defined in (2.13) and (5.14), respectively. Thus, from (5.41) and the differential identity (3.5), we have

$$\begin{aligned} \frac{d}{dt}h_n = & -\frac{\pi i \alpha}{2^{2(n-1)}}D_t(\infty)^2 \left( C_{12}^2(E_0)_{21}(E_0)_{22} \frac{1}{\sqrt{2(t-1)}} \right. \\ & \left. + C_{12}C_{11}((E_0)_{12}(E_0)_{21} + (E_0)_{11}(E_0)_{22}) + C_{11}^2(E_0)_{11}(E_0)_{12}\sqrt{2(t-1)} \right), \end{aligned} \quad (5.42)$$

where  $C_{i,j}$  stands for the  $(i, j)$  entry of the matrix

$$C = \left( I + O\left(\frac{1}{n}\right) \right) M_1 \sum_{k=0}^{\infty} C_k(t-1)^{\frac{k}{2}}; \quad (5.43)$$

see (5.14) for the constant matrices  $M_1$ ,  $C_0$  and  $C_1$ . In view of (2.13), (2.15) and Proposition 1, we may write

$$(E_0)_{21}(E_0)_{22} = \frac{2i}{\alpha}(\sigma - su)', \quad (E_0)_{11}(E_0)_{12} = \frac{i}{2\alpha}(\sigma + su)', \quad (5.44)$$

and

$$(E_0)_{12}(E_0)_{21} = \frac{1}{2\alpha} \left( su - \beta + \frac{1}{2} - \alpha \right), \quad (E_0)_{11}(E_0)_{22} = \frac{1}{2\alpha} \left( su - \beta + \frac{1}{2} + \alpha \right). \quad (5.45)$$

Now, substituting (5.43), (5.44) and (5.45) into (5.42), we obtain

$$\begin{aligned} \frac{dh_n}{dt} = & -\frac{2\sqrt{2}\pi D_t(\infty)^2}{2^{2n}} \left\{ \frac{(\sigma - su)'}{\sqrt{t-1}} \left( 1 + O\left(\frac{1}{n}\right) + O(t-1) \right) \right. \\ & \left. + (\sigma + su)'O(\sqrt{t-1}) \right. \\ & \left. - \left( su - \beta + \frac{1}{2} \right) \left( \frac{1}{\sqrt{2}} + d_1 + O\left(\frac{1}{n}\right) + O(\sqrt{t-1}) \right) \right\}, \end{aligned} \quad (5.46)$$

where  $d_1$  is defined in (5.11) and the error term  $O(\frac{1}{n})$  is uniform for  $t \in (1, d]$ .

From (2.27) and (2.28), we can derive the estimates of the integrals

$$\int_1^t (su - \beta + \frac{1}{2})\sqrt{t-1}dt = O\left(\frac{t-1}{n}\right) + O\left((t-1)^{\frac{3}{2}}\right) \quad (5.47)$$

and

$$\int_1^t (\sigma + su)'\sqrt{t-1}dt = o\left(\frac{t-1}{n}\right) + O\left((t-1)^{\frac{3}{2}}\right). \quad (5.48)$$

Now, in view of (5.7), (2.27) and (2.28), and integrating by parts once, we have

$$\begin{aligned} \int_1^t \frac{D_t(\infty)^2(\sigma - su)'dt}{\sqrt{t-1}} &= 2D_t(\infty)^2 \left( \frac{\sigma}{s} - u \right) \left( \sqrt{t-1} + O\left((t-1)^{\frac{3}{2}}\right) \right) \\ &\quad - \frac{\sqrt{2}\alpha}{8n^2} \int_0^s D_t(\infty)^2(\sigma - su)ds + O\left(\frac{t-1}{n}\right) \\ &\quad - \frac{\sqrt{2}D_1(\infty)^2}{4n} \left( \frac{1}{4} - (\alpha + \beta)^2 \right) + O\left((t-1)^{\frac{3}{2}}\right). \end{aligned} \quad (5.49)$$

Then, integrating both sides of (5.46) and using the estimates (5.47)-(5.49), we obtain

$$\begin{aligned} h_n(t) &= h_n(1) - \frac{\pi D_1(\infty)^2}{2^{2n}} \frac{4(\alpha + \beta)^2 - 1}{4n} - \frac{\pi D_t(\infty)^2}{2^{2n}} \left\{ 4\sqrt{2} \left( \frac{\sigma}{s} - u \right) \sqrt{t-1} \right. \\ &\quad \left. + I(s) \frac{1}{n^2} + O\left(\frac{\sqrt{t-1}}{n}\right) + O\left((t-1)^{\frac{3}{2}}\right) \right\}, \end{aligned} \quad (5.50)$$

where

$$I(s) = -\frac{\sqrt{2}\alpha}{8D_t(\infty)^2} \int_0^s D_t(\infty)^2 \left\{ (\sigma - su) + \frac{1 + \sqrt{2}d_1}{4\alpha} \left( su - \beta + \frac{1}{2} \right) s \right\} ds. \quad (5.51)$$

By (2.27) and (2.28), we get the estimate

$$I(s) = O(s^2) + O(s). \quad (5.52)$$

To determined  $h_n(1)$ , we use a result from [18, Thm. 1.6], that is,

$$h_n(1) = \frac{\pi}{2^{2n}} D_1(\infty)^2 \left( 1 + \frac{4(\alpha + \beta)^2 - 1}{4n} + c_n \right), \quad c_n = O\left(\frac{1}{n^2}\right). \quad (5.53)$$

Substituting (5.52) and (5.53) into (5.50), and noting that the relation between  $D_1(\infty)$  and  $D_t(\infty)$  (5.7), we have

$$\begin{aligned} h_n(t) &= \frac{\pi}{2^{2n}} D_t(\infty)^2 \left\{ 1 - \left[ 4\sqrt{2} \left( \frac{\sigma}{s} - u \right) + 2\alpha\sqrt{2} \right] \sqrt{t-1} + O\left((t-1)^{\frac{3}{2}}\right) \right. \\ &\quad \left. + O\left(\frac{s}{n^2}\right) + c_n(t + \sqrt{t^2 - 1})^{-2\alpha} \right\}, \end{aligned} \quad (5.54)$$

where  $c_n$  is independent of  $t$  and  $c_n = O(\frac{1}{n^2})$ . It follows from Proposition 2 that

$$\sigma - su = 4\sigma_{\text{JM}} \left( \frac{s^2}{16} \right) - \frac{s^2}{16} - (\alpha + \beta)^2 + \frac{1}{4}. \quad (5.55)$$

Since  $h_n = \gamma_n^{-2}$ , we obtain the asymptotic approximation of the leading coefficient as in (1.11). Thus completing the proof of Theorem 2.



### 5.3 Proof of Theorem 3

To approximate the recurrence coefficients, we recall their relation with the leading coefficients

$$b_{n-1}^2 = \frac{\gamma_{n-1}^2}{\gamma_n^2}. \quad (5.56)$$

Let

$$\Lambda_1 = \frac{\alpha}{2}s + \sigma - us, \quad (5.57)$$

then  $\Lambda_1$  is analytic for  $s \in (0, \infty)$  and

$$\Lambda_1(s) = \frac{1}{4} - (\alpha + \beta)^2 + O(s^l) \quad \text{as } s \rightarrow 0^+ \quad (5.58)$$

with  $l = \min(1, 2(1 + \alpha + \beta))$ , and

$$\Lambda_1(s) = \frac{1}{4} - \beta^2 + \sum_{k=1}^{\infty} \frac{l_k}{s^k} \quad \text{as } s \rightarrow +\infty, \quad (5.59)$$

where  $l_k$  are constant coefficients. Then it follows from (5.57), (5.58) and (5.59) that

$$\Lambda_1(s - s/n) = \Lambda_1(s) - \frac{s}{n}\Lambda_1'(s) + O\left(\frac{s^2}{n^2}\right), \quad (5.60)$$

where the error term is uniform for  $s \geq 0$ . From (5.60), and using the fact that  $s \sim 4\sqrt{2}n\sqrt{t-1}$  as  $t \rightarrow 1$ , we have

$$\frac{\Lambda_1(s - \frac{s}{n})}{s - \frac{s}{n}} = \frac{\Lambda_1}{s} - 4\sqrt{2}\left(\frac{\Lambda_1}{s}\right)' \sqrt{t-1} + O\left(\frac{s}{n^2}\right). \quad (5.61)$$

Using the expression of  $I(s)$  in (5.51), and in view of (2.27) and (2.28), we get

$$\frac{I(s - \frac{s}{n})}{(n-1)^2} = \frac{I(s)}{n^2} + O\left(\frac{t-1}{n}\right). \quad (5.62)$$

Then, a combination of (5.50), (5.53), (5.61) and (5.62) gives the following asymptotic formula for  $h_{n-1}$

$$\begin{aligned} h_{n-1}(t) = & \frac{4\pi D_t(\infty)^2}{2^{2n}} \left\{ 1 - 4\sqrt{2} \frac{\Lambda_1(s)}{s} \sqrt{t-1} + 32 \left(\frac{\Lambda_1}{s}\right)' (t-1) + \frac{I(s)}{n^2} \right. \\ & \left. + c_n \left(t + \sqrt{t^2 - 1}\right)^{-2\alpha} + O\left(\frac{t-1}{n}\right) + O\left((t-1)^{\frac{3}{2}}\right) + O\left(\frac{\sqrt{t-1}}{n^2}\right) \right\}. \end{aligned} \quad (5.63)$$

From (5.50), (5.63) and (5.56), we obtain the asymptotic approximation of the recurrence coefficients stated in (1.12).

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# References

- [1] E. Basor, Y. Chen and N. Haq, Asymptotics of determinants of Hankel matrices via non-linear difference equations, arXiv:1401.2073v1.
- [2] L. Brightmore, F. Mezzadri and M.Y. Mo, A matrix model with a singular weight and Painlevé III, *Comm. Math. Phys.*, (2014) doi:10.1007/s00220-014-2076-z.
- [3] Y. Chen and L. Zhang, Painlevé VI and the Unitary Jacobi Ensembles, *Stud. Appl. Math.*, **125** (2010), 91–112.
- [4] T. Claeys, A.B.J. Kuijlaars and M. Vanlessen, Multi-critical unitary random matrix ensembles and the general Painlevé II equation, *Ann. of Math.*, **168**(2008), 601–641.
- [5] T. Claeys, A. Its and I. Krasovsky, Emergence of a singularity for Toeplitz determinants and Painlevé V, *Duke Math. J.*, **160** (2011), 207–262.
- [6] T. Claeys and I. Krasovsky, Toeplitz determinants with merging singularities. arXiv preprint arXiv:1403.3639.
- [7] D. Dai and A.B.J. Kuijlaars, Painlevé IV Asymptotics for Orthogonal Polynomials with Respect to a Modified Laguerre Weight, *Stud. Appl. Math.*, **122** (2009), 29–83.
- [8] P. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant Lecture Notes 3, New York University, 1999.
- [9] P. Deift, A. Its and I. Krasovsky, Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities, *Ann. of Math.*, **174** (2011), 1243–1299.
- [10] P. Deift, A. Its and I. Krasovsky, Toeplitz matrices Toeplitz determinants under the impetus of the Ising model. Some history and some recent results, *Comm. Pure Appl. Math.*, **66** (2013), 1360–1438.
- [11] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights

- and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.*, **52** (1999), 1335–1425.
- [12] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.*, **52** (1999), 1491–1552.
  - [13] A.S. Fokas, A.R. Its and A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.*, **147** (1992), 395–430.
  - [14] P.J. Forrester and N.S. Witte, Application of the  $\tau$ -function theory of Painlevé equations to random matrices: PV, PIII, the LUE, JUE and CUE, *Comm. Pure Appl. Math.*, **55** (2002), 679–727.
  - [15] P.J. Forrester and N.S. Witte, Boundary conditions associated with the Painlevé III' and V evaluation of some random matrix average, *J. Phys. A*, **39** (2006), 8983–8995.
  - [16] A.R. Its, A.B.J. Kuijlaars and J. Östensson, Critical edge behavior in unitary random matrix ensembles and the thirty fourth Painlevé transcendent, *Int. Math. Res. Not.*, **2008** (2008), article ID rnn017:67 pp.
  - [17] M. Jimbo, Monodromy problem and the boundary condition for some Painlevé equations. *Publ. Res. Inst. Math. Sci.*, **18** (1982), 1137–1161.
  - [18] A.B.J. Kuijlaars, K.T.-R. McLaughlin, W. Van Assche and M. Vanlessen, The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on  $[-1, 1]$ , *Adv. Math.*, **188** (2004), 337–398.
  - [19] A.B.J. Kuijlaars and M. Vanlessen, Universality for eigenvalue correlations from the modified Jacobi unitary ensemble, *Int. Math. Res. Not.*, **2002** (2002), 1575–1600.
  - [20] M.L. Mehta, *Random matrices*, 3rd ed., Elsevier/Academic Press, Amsterdam, 2004.
  - [21] G. Szegő, *Orthogonal Polynomials*, 4th edition, AMS Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence R.I., 1975.
  - [22] C. Tracy and H. Widom, Level spacing distributions and the Bessel kernel, *Comm. Math. Phys.*, **161** (1994), 289–309.
  - [23] M. Vanlessen, Strong asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight, *J. Approx. Theory*, **125** (2003), 198–237.
  - [24] S.-X. Xu, D. Dai and Y.-Q. Zhao, Critical edge behavior and the Bessel to Airy transition in the singularly perturbed Laguerre unitary ensemble, *Comm. Math. Phys.*, **332** (2014), 1257–1296.

- [25] S.-X. Xu, D. Dai and Y.-Q. Zhao, Painlevé III asymptotics of Hankel determinants for a singularly perturbed Laguerre weight, *J. Approx. Theory*, **192** (2015), 1–18.
- [26] S.-X. Xu and Y.-Q. Zhao, Critical edge behavior in the modified Jacobi ensemble and the Painlevé equation, arXiv:1404.5105v2.
- [27] S.-X. Xu and Y.-Q. Zhao, Painlevé XXXIV asymptotics of orthogonal polynomials for the Gaussian weight with a jump at the edge, *Stud. Appl. Math.*, **127** (2011), 67–105.
- [28] S.-X. Xu and Y.-Q. Zhao, Critical edge behavior in the modified Jacobi ensemble and the Painlevé V transcendents, *J. Math. Phys.*, **54** (2013), 083304.